Comparison of Several Methods for Calculating Vibration Mode Shape Derivatives

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Four methods for the calculation of derivatives of vibration mode shapes (eigenvectors) with respect to design parameters are described. These are finite-difference method, modal method, a modified modal method and Nelson's method. The methods are implemented in a general-purpose commercial finite-element program and applied to the following test problems: a cantilever beam and a stiffened cylinder with a cutout. Design variables are a beam tip mass, a beam root height, and specific dimensions of the cylinder model. The methods are compared on the basis of central processor (CP) seconds required to obtain the derivatives, and two of the methods are also evaluated for the rapidity of convergence. Data is presented showing the amount of CP time used to compute the first four eigenvector derivatives for each example problem. A scalar measure of the error in the mode shape derivative is defined, and numerical results illustrating the rapidity of convergence of the approximate derivative to the exact derivative are presented. Results indicate an advantage in using Nelson's method because this method is exact and requires less CP time, especially when derivatives with respect to several design variables are computed.

Nomenclature

A_{ijk}	= coefficient in Eq. (6)
$\bar{A}_{ijk}$	= coefficient in Eq. (11)
h	= height of beam root element, Fig. 1
H	= channel height, Fig. 2
$[K]$	= stiffness matrix
$[M]$	= mass matrix
m	= concentrated mass at the beam tip
N	= number of system eigenvectors.
s	= time in seconds
T^1,T^2	= channel thickness, Fig. 2
V_i	= ith design variable
W	= width of channel, Fig. 2
x,y,z	= Cartesian coordinates
$\{\delta\}$	= difference between approximate and exact derivative, Eq. (19)
ε	= error measure, Eq. (17)
γ	= mode shape derivative magnitude, Eq. (18)
λ_j	= jth eigenvalue
$\{\phi\}_j$	= jth eigenvector (mode shape)
$\partial/\partial V_i$	= derivative with respect V_i
$\lvert\ \rvert$	= vector magnitude

Introduction

SENSITIVITY analysis, the study of changes in system response with respect to parameter variations, is being used in a variety of engineering disciplines ranging from automatic control theory to the analysis of large-scale physiological systems.[1] Some of the areas where sensitivity analysis has been applied include 1) system identification, 2) development of insensitive control systems, 3) use in gradient-based mathematical programming methods, 4) approximation of system response to a change in a system parameter, and 5) assessment of design changes on system performance.[2] Recent advances have been made in calculating sensitivity derivatives for static and transient response quantities,[2,3] eigenvalues and eigenvectors,[2,4–8] and optimum structural designs with respect to problem parameters.[9]

Derivatives of vibration mode shapes (eigenvectors) with respect to design variables are particularly useful in certain analysis and design applications, e.g., approximating a new vibration mode shape due to a perturbation in a design variable, determining the effect of design changes on the dynamic behavior of a structure,[2] and tailoring mode shapes to minimize displacements at certain points on a structure. In contrast to computing eigenvalue derivatives where preferred methods exist, there are a number of different methods for calculating mode shape derivatives. The different methods seek to overcome the practical difficulty of solving a singular matrix equation.[2,4–8]

Methods for calculating mode shape derivatives include finite-difference method, modal method,[4] a modified modal method,[5] and Nelson's method.[6] The finite-difference method uses a difference formula to numerically approximate the derivative, which requires calculating the eigenvector at a nominal and at least one perturbed design point. This method is sensitive to roundoff and truncation errors associated with the step size used. The modal method approximates the mode shape derivative as a linear combination of mode shapes. This method can be computationally expensive if a large number of modes are needed to accurately represent the mode shape derivative. The modified modal method was developed to reduce the number of modes needed to represent the derivative by including an additional term in the linear combination of the system mode shapes. Nelson's method is an exact analytical method for calculating mode shape derivatives.

The purpose of the study described in this paper is to evaluate the methods described previously on the basis of central processor (CP) time and to compare the modal and the modified modal method on the basis of rapidity of

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convergence. Finite-element models of a cantilever beam and a stiffened cylinder with a cutout are used as test problems in the study. The derivative methods are implemented in the Engineering Analysis Language System[10] using methodology similar to that presented in Refs. 11 and 12.

Governing Equations

The matrix representation of the free vibration eigenvalue problem is

$$([K]-\lambda_j[M])\{\phi\}_j=0 \tag{1}$$

In this study, it is assumed that $\{\phi\}_j$ is normalized such that

$$\{\phi\}_j^T[M]\{\phi\}_j=1.0 \tag{2}$$

The expression for the eigenvalue derivative with respect to the ith design variable is (see, for example, Ref. 2)

$$\frac{\partial\lambda_j}{\partial V_i}=\{\phi\}_j^T\frac{\partial[K]}{\partial V_i}\{\phi\}_j-\lambda_j\{\phi\}_j^T\frac{\partial[M]}{\partial V_i}\{\phi\}_j \tag{3}$$

Given an eigensolution corresponding to a nominal value of a design variable V_i, and assuming that $(\partial[K]/\partial V_i)$ and $(\partial[M]/\partial V_i)$ can be calculated, the solution of Eq. (3) is straightforward.

Differentiating Eq. (1) with respect to a design variable V_i, the governing equation for eigenvector derivatives is

$$([K]-\lambda_j[M])\frac{\partial\{\phi\}_j}{\partial V_i}=\frac{\partial\lambda_j}{\partial V_i}[M]\{\phi\}_j-\frac{\partial[K]}{\partial V_i}\{\phi\}_j+\lambda_j\frac{\partial[M]}{\partial V_i}\{\phi\}_j \tag{4}$$

A direct solution of Eq. (4) is not possible since $([K]-\lambda_j[M])$ is singular. The various methods evaluated in this paper all seek, in different ways, to overcome the complication associated with this singularity.

Equations (3) and (4) are only valid for the case where Eq. (1) has distinct eigenvalues. Derivative expressions for the case of repeated eigenvalues have recently been presented in Refs. 7 and 8 but are not considered in this paper.

Description of the Methods

Four methods for calculating eigenvector derivatives $(\partial\{\phi\}_j/\partial V_i)$ are described. Every method, except the finite-difference method, requires the mass matrix and stiffness matrix derivatives $(\partial[M]/\partial V_i)$ and $(\partial[K]/\partial V_i)$, respectively. In the work described in this paper, the mass matrix and stiffness matrix derivatives are calculated by a forward finite-difference formula. It should be noted that exact analytical derivatives for the mass and stiffness matrices can be obtained for bar elements if the cross-sectional area is the design variable and, for shear and membrane panels, if the panel thickness is the design variable.

Finite-Difference Method

In the finite-difference method, Eq. (1) is solved for $\{\phi\}_{j\,\text{old}}$, the ith design variable is perturbed by ΔV_i, and a new eigenvector $\{\phi\}_{j\,\text{new}}$ is obtained by solving Eq. (1) again, where V_i new $= V_{i\,\text{old}}+\Delta V_i$. The derivative is approximated by the expression

$$\frac{\partial\{\phi\}_j}{\partial V_i}=\frac{\{\phi\}_{j\,\text{new}}-\{\phi\}_{j\,\text{old}}}{\Delta V_i} \tag{5}$$

To reduce numerical errors associated with Eq. (5), attention should be paid to the step size ΔV_i. For this paper $\Delta V_i=0.01V_i$. An algorithm for determining the optimum step size has been developed to further reduce numerical errors and is described in Ref. 13.

Modal Method

The modal method expresses the derivative of an eigenvector as a series expansion of the system eigenvectors. The approximate derivative is expressed as

$$\frac{\partial\{\phi\}_j}{\partial V_i}=\sum_{k=1}^{N}A_{ijk}\{\phi\}_k \tag{6}$$

where the coefficients A_{ijk} are calculated using

$$A_{ijk}=\{\phi\}_k^T\left(\frac{\partial[K]}{\partial V_i}-\lambda_j\frac{\partial[M]}{\partial V_i}\right)\{\phi\}_j\Big/(\lambda_j-\lambda_k)\ \text{for}\ k\neq j \tag{7}$$

For $k=j$, Eq. (2) is differentiated to obtain

$$2\{\phi\}_j^T[M]\frac{\partial\{\phi\}_j}{\partial V_i}+\{\phi\}_j^T\frac{\partial[M]}{\partial V_i}\{\phi\}_j=0 \tag{8}$$

The expression for $(\partial\{\phi\}_j/\partial V_i)$ from Eq. (6) is substituted into Eq. (8), and using the orthogonality condition [Eq. (2)], the coefficient A_{ijk} is obtained:

$$A_{ijk}=-\frac{1}{2}\{\phi\}_j^T\frac{\partial[M]}{\partial V_i}\{\phi\}_j\quad\text{for}\quad k=j \tag{9}$$

Modified Modal Method

The modified modal method uses a pseudostatic solution of Eq. (4) as an initial approximation to the mode shape derivative. This is similar in principle to the mode-acceleration method used in transient structural analysis.[14] Equation (4) is solved by neglecting the quantity $\lambda_j[M](\partial\{\phi\}_j/\partial V_i)$ and obtaining the pseudostatic solution for $(\partial\{\phi\}_j/\partial V_i)_s$, which is

$$\left(\frac{\partial\{\phi\}_j}{\partial V_i}\right)_s=[K]^{-1}\left(\frac{\partial\lambda_j}{\partial V_i}[M]-\frac{\partial[K]}{\partial V_i}+\lambda_j\frac{\partial[M]}{\partial V_i}\right)\{\phi\}_j \tag{10}$$

This pseudostatic solution is added to Eq. (6) to obtain

$$\frac{\partial\{\phi\}_j}{\partial V_i}=\left(\frac{\partial\{\phi\}_j}{\partial V_i}\right)_s+\sum_{k=1}^{N}\bar{A}_{ijk}\{\phi\}_k \tag{11}$$

where $\bar{A}_{ijk}$ are coefficients for the modified modal method.

To obtain the coefficients $\bar{A}_{ijk}$, Eq. (11) is substituted into Eq. (4), and the result is premultiplied by $\{\phi\}_k^T$. When simplified, this result becomes

$$\bar{A}_{ijk}=\lambda_j\{\phi\}_k^T\left(\frac{\partial[K]}{\partial V_i}-\lambda_j\frac{\partial[M]}{\partial V_i}\right)\{\phi\}_j\Big/\lambda_k(\lambda_j-\lambda_k)\quad\text{for}\ k\neq j \tag{12}$$

$$\bar{A}_{ijk}=-\frac{1}{2}\{\phi\}_j^T\frac{\partial[M]}{\partial V_i}\{\phi\}_j\quad\text{for}\quad k=j \tag{13}$$

The relative convergence of the modified modal method vs the modal method for a given number of eigenvectors can be anticipated by dividing Eq. (12) by Eq. (7):

$$\bar{A}_{ijk}/A_{ijk}=\lambda_j/\lambda_k \tag{14}$$

Assuming that to calculate $(\partial\{\phi\}_j/\partial V_i)$ accurately j modes or more are needed; then for $k>j$, $\bar{A}_{ijk}$ is smaller than A_{ijk}, and Eq. (11) will converge faster than Eq. (6).

Nelson's Method

Nelson's method obtains an exact solution to Eq. (4). This method expresses the eigenvector derivative in terms of a particular solution $\{P\}$ and a complementary solution $C\{\phi\}_j$, where C is an undetermined coefficient, thus

$$\frac{\partial\{\phi\}_j}{\partial V_i}=\{P\}+C\{\phi\}_j \tag{15}$$

The particular solution is found by identifying the component of the eigenvector $\{\phi\}_j$ with the largest absolute value and constraining the derivative of that component to zero. The value of C in Eq. (15) is found by substituting Eq. (15) into Eq. (8) and simplifying to obtain

$$C = -\{\phi\}_j^T[M]\{P\} - \frac{1}{2}\{\phi\}_j^T \frac{\partial[M]}{\partial V_i}\{\phi\}_j \tag{16}$$

Implementation of the Methods

The methods described previously have been implemented in the Engineering Analysis Language (EAL) System, which is a general-purpose commercial finite-element program. This program consists of a series of processors that communicate with each other through a data base that is comprised of one or more libraries of data sets. The processors perform specific functions such as building the data sets that describe the model, defining element stiffness matrices, factoring the global stiffness matrix, etc. Each data set contains information describing the finite-element model such as the joint locations, the material or section properties, the element connectivities, and analysis results such as eigenvectors and stresses. An Executive Control System provides the user with control of the sequence of execution of the processors and the ability to manipulate the input and output for each processor through a programming-like language. The programming-like feature of EAL was used extensively to implement each of the mode shape derivative methods described in this paper.

Example Problems

The four methods described previously for calculating mode shape derivatives have been applied to the following example problems: a cantilever beam and a stiffened cylinder with a cutout.

Cantilever Beam

The first example problem is a 12-node cantilever beam (Fig. 1) that is modeled using 11 rectangular cross-section beam elements. There are two degrees of freedom (DOF) at each node corresponding to translation in the y-direction and rotation about an axis normal to the x-y plane. Both degrees of freedom at the first node are constrained to zero, yielding a total of 22 DOF for the model. The derivatives of the first four mode shapes (eigenvectors) are calculated with respect to a design variable that is the height h of the root element. The same model is used to obtain the first four eigenvector derivatives with respect to a design variable, which is a concentrated mass m at the tip of the beam. This relatively simple model is used to verify the implementation of each method with the EAL finite-element program and to provide an inexpensive model for using all of the system eigenvectors with the modified modal and modal methods.

Stiffened Cylinder with Cutout

The second example problem is a stiffened cylinder with a cutout (Fig. 2). The model has 80 nodes and is composed of 76 channel-type beam elements, 56 quadrilateral membrane elements, and 58 rod elements. The 16 equally spaced stringers are modeled by rod elements, and the five equally spaced rings are modeled with the channel-type beam elements. Translation at one edge is constrained to zero, and the model has a total of 352 DOF. A total of nine design variables are considered for the stiffened cylinder: 1) thickness of the membrane elements, 2) width W of the channel section, 3) channel flange thickness $T1$, 4) channel height H, 5) channel web thickness $T2$, and, 6–9) the areas of four selected stringers (rods).

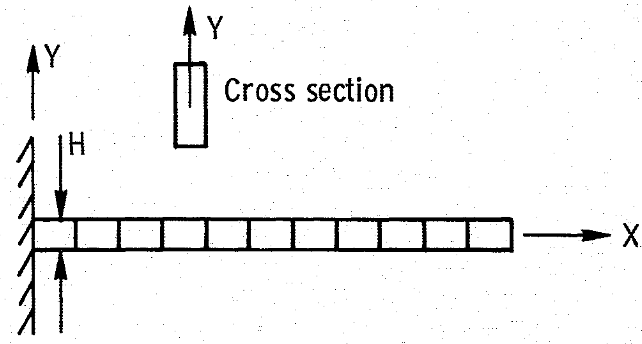


Fig. 1 Cantilever beam with height h as the design variable.

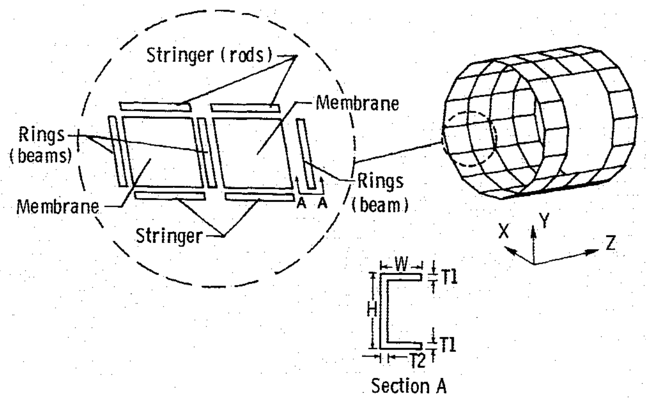
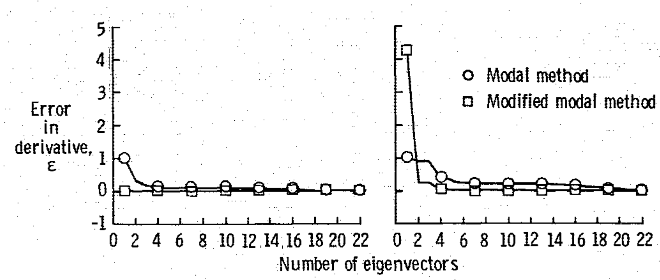


Fig. 2 Stiffened cylinder with a cutout.



a) First mode b) Third mode

Fig. 3 Convergence of mode shape derivatives with respect to root element height h for cantilever beam.

Results and Discussion

Convergence Measure for Modal Methods

Since the modal method and the modified modal method use a subset of the total number of system eigenvectors in the approximation of a mode shape derivative, a scalar measure of the error in the derivative was developed to measure convergence. The error measure is defined as

$$\epsilon = \frac{(\{\delta\}^T\{\delta\})^{1/2}}{\gamma} \tag{17}$$

where

$$\gamma = \left|\frac{\partial\{\phi\}_j}{\partial V_i}\right|_{\text{Nelson}} \tag{18}$$

$$\{\delta\} = \left(\frac{\partial\{\phi\}_j}{\partial V_i}\bigg|_{\text{approx}} - \frac{\partial\{\phi\}_j}{\partial V_i}\bigg|_{\text{Nelson}}\right) \tag{19}$$

The error measure provides a method of evaluating the convergence of the derivative calculated by the modal or the modified modal method to Nelson's method for each eigenvector used in the summation [Eqs. (6) and (11)]. Convergence of each mode shape derivative is defined for the modal and modified modal methods when the value of the error parameter ε [Eq. (17)] is less than 0.01.

Convergence of Modal Methods for the Cantilever Beam

The convergence of the mode shape derivative using the modal and the modified modal methods for the cantilever beam with the height h as the design variable is shown in Fig. 3. Figures 3a and 3b show the convergence of the first and third mode shape derivatives, respectively. The modified modal method requires two and six eigenvectors to represent the first and third mode shape derivatives, respectively, whereas the modal method requires 21 modes for each derivative. A similar convergence history (not shown) is obtained for the second and fourth mode shape derivatives. The modified modal method converges faster than the modal method when at least as many modes are used in the summation as the number of the mode being differentiated. This means that the number of modes used in the summation N is equal to or greater than the value for j in the term $(\partial\{\phi\}_j/\partial V_i)$. In Fig. 4, the y-component of the first mode shape derivative is plotted. The values of the derivative are normalized such that the maximum value of the derivative is equal to 1.0. The shape of the derivative is similar to the second mode shape of the cantilever beam and indicates why at least the first two mode shapes must be used in Eq. (11) to represent the derivative with the modified modal method.

The convergence of the modal and the modified modal methods for the derivatives of the first and third mode shapes of the cantilever beam with respect to a concentrated mass m is shown in Figs. 5a and 5b. A similar convergence history (not shown) is obtained for the second and fourth mode shape derivatives. Both methods obtain mode shape derivatives very close to the exact solution (Nelson's method) using fewer modes than needed for the cantilever beam with the height h as the design variable. The modified modal method requires one and four eigenvectors to obtain the first and third mode shape derivatives, respectively, whereas the modal method requires two and six eigenvectors, respectively. In Fig. 6, the y-component of the first mode shape derivative is shown. The derivative is normalized as described for the cantilever beam where the design variable is the height of the root element. The shape of the first eigenvector derivative is similar to the first mode shape of a cantilever beam and indicates why only one mode shape is needed to represent the derivative with the modified modal method.

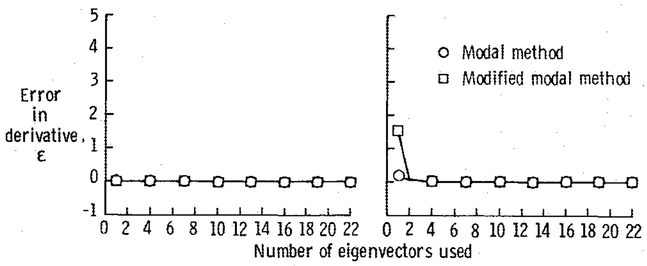
Convergence of Modal Methods for Stiffened Cylinder with Cutout

The derivatives of the first four mode shapes are obtained for the stiffened cylinder using the first two design variables. In Figs. 7a and 7b, the convergence of the derivatives of the first and third mode shapes, respectively, is shown for the first design variable. The modified modal method achieves convergence using six and four modes for the derivatives of the first and third mode shapes, respectively, whereas the modal method did not converge for the first derivative using as many as 20 modes. Convergence is obtained with the modal method for the third mode shape derivative using nine eigenvectors. A similar convergence history (not shown) is obtained for the second and fourth mode shape derivatives with respect to the second design variable. For both design variables, the modified modal method converges more rapidly to the exact derivative than the modal method.

Computational Performance of the Methods

In Table 1, the CP seconds required for calculating the derivatives of first four mode shapes are shown for the cantilever beam and the stiffened cylinder. The times are categorized by the major solution steps, such as assembling the mass matrix or obtaining an eigensolution. A CDC CYBER 855 computer with the Network Operating System, level 2.3, is used to obtain the results. Only the first four mode shapes are calculated in each of the examples shown for the finite difference and Nelson's methods. When the modified modal or modal methods are used, all 22 eigenvectors are obtained for the cantilever beam examples, and 20 mode shapes are calculated with the stiffened cylinder. However, the modified modal method required substantially fewer than these numbers of modes for convergence, and the modal method often required fewer modes for convergence. The performance of the modal methods would be significantly improved if the number of modes needed to represent a mode shape derivative was known a priori, since the CP time required to calculate 22 and 20 modes is a significant portion of the total CP time used in each method.

For the first and third cases in Table 1, Nelson's method requires the least amount of CP time for computing the first four eigenvector derivatives, followed closely by the finite-difference method. Nelson's method did not outperform the finite-difference method for the second example problem because the time needed to obtain the new eigenvectors and calculate the derivatives is less than the total time needed to calculate the derivatives with Nelson's method. The stiffened cylinder model clearly shows the advantage of Nelson's



a) First mode b) Third mode

Fig. 5 Convergence of mode shape derivatives with respect to tip mass *m* for cantilever beam.

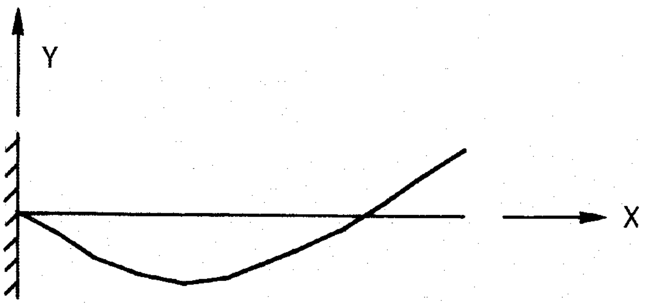


Fig. 4 Derivative of the first mode shape with respect to height *h* of the cantilever beam.

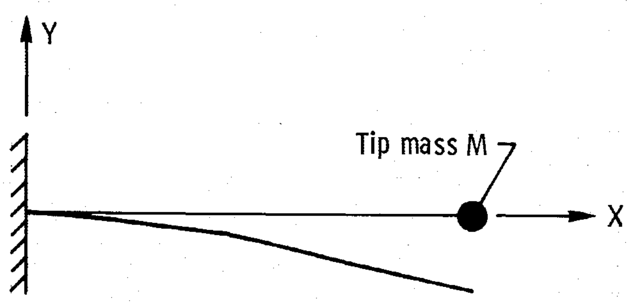
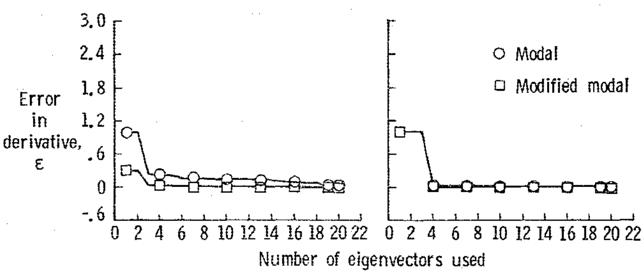


Fig. 6 Derivative of first mode shape with respect to tip mass *m* for cantilever beam.

Table 1 Central processor seconds required to calculate the first four mode shape derivatives

Method	Case 1 Cantilever beam, DV^a = height	Case 2 Cantilever beam, DV^a = mass	Case 3 Stiffened cylinder, DV^a = membrane thickness
Finite difference			
assemble $[K]$, $[M]$	2.0	1.8	14.8
eigensolution $[K - \lambda M]\{\phi\}$	16.4	13.7	275.8
perturb $[K]$, $[M]$	1.4	0.2	16.2
eigensolution $[K - \lambda M]\{\phi\}$	6.7	2.3	73.4
obtain derivatives $\partial\{\phi\}/\partial V$	1.4	1.4	1.6
Total	27.9	19.4	381.8
Nelson method			
assemble $[K]$, $[M]$	1.8	1.9	15.0
eigensolution $[K - \lambda M]\{\phi\}$	16.2	13.7	275.3
perturb $[K]$, $[M]$	1.4	0.2	12.0
calculate $(\partial[K]/\partial V)$, $(\partial[M]/\partial V)$	0.5	0.3	0.3
obtain derivatives $(\partial\{\phi\}/\partial V)$	7.0	7.8	34.5
Total	26.9	23.9	337.1
Modified modal method			
assemble $[K]$, $[M]$	1.8	1.9	14.4
eigensolution $[K - \lambda M]\{\phi\}$	32.8	30.8	1529.6
perturb $[K]$, $[M]$	1.3	0.3	12.0
calculate $(\partial[K]/\partial V)$, $(\partial[M]/\partial V)$	0.4	0.1	0.3
obtain derivatives $(\partial\{\phi\}/\partial V)$	39.3	28.1	59.1
Total	75.6	61.2	1615.4
Modal method			
assemble $[K]$, $[M]$	1.5	1.9	14.5
eigensolution $[K - \lambda M]\{\phi\}$	28.9	31.0	1529.0
perturb $[K]$, $[M]$	1.2	0.3	11.9
calculate $(\partial[K]/\partial V)$, $(\partial[M]/\partial V)$	0.5	0.2	0.4
obtain derivatives $(\partial\{\phi\}/\partial V)$	114.8	33.8	113.3
Total	146.9	61.2	1669.1

[a] DV = design variable.



a) First mode b) Third mode

Fig. 7 Convergence of mode shape derivatives with respect to membrane thickness for stiffened cylinder.

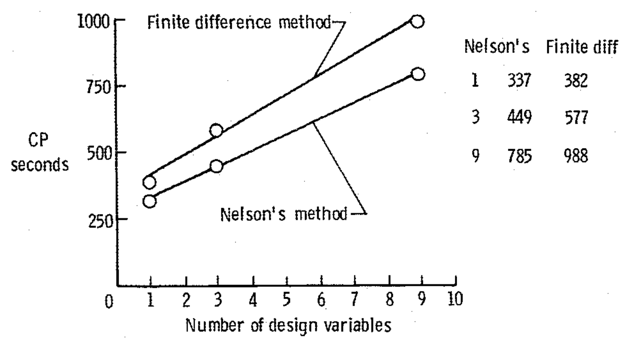


Fig. 8 CP time required to compute four mode shape derivatives for stiffened cylinder.

method over the finite-difference method. In this case, the mode shape derivatives are computed in less than 50% of the CP time needed to obtain the first four eigenvectors for the perturbed design variable.

In Fig. 8, the CP seconds required to obtain the first four mode shape derivatives as a function of the number of design variables is shown for the stiffened cylinder model, using Nelson's method and the finite-difference method. For the stiffened cylinder model in Table 1, it can be seen that the CP time required to perturb the model and calculate new eigenvectors in the finite-difference method is greater than the time required to compute $(\partial[K]/\partial V_i)$ and $(\partial[M]/\partial V_i)$, and to obtain the derivatives using Nelson's method. Thus, Nelson's method will become even more attractive for problems having a large number of design variables.

If the number of modes N needed to obtain one mode shape derivative could be determined before performing an eigensolution, then the modified modal method would be competitive with Nelson's method, at least for the first mode shape derivative. To illustrate this point, the CP time required to compute the derivative of the first mode shape in the first example problem shown in Table 1 is 2.4 s, and only the first two eigenvectors are needed to accurately represent the derivative. Estimating the CP time needed to obtain the first two modes at 16.0 s (which is a conservative estimate since 16.2 s are needed for four modes), then the total time for the calculation of the derivative would be 21.9 s instead of the 75.6 s shown. Also, as the problem size becomes larger, the cost of the system matrix factorization in Nelson's method would become proportionally more costly relative to the matrix operations in the modal methods. This can be seen in Table 1 by comparing the CP times required to compute the derivatives for the cantilever beam and the stiffened cylinder.

Concluding Remarks

This paper reviewed and compared four methods for calculating vibration mode shape derivatives with respect to design variables. They were the finite-difference method, the modal method, a modified modal method, and Nelson's method. All four methods were implemented in a general-purpose finite-element program and applied to the following example problems: a cantilever beam and a stiffened cylinder with a cutout. A scalar measure of the error in the derivative was defined for determining convergence of the mode shape derivatives obtained using the modal and the modified modal methods. The time in CP seconds required to calculate the first four mode shape derivatives using each method was presented. Also, numerical results showing the convergence of the first and third mode shape derivatives using the modal and the modified modal methods as a function of the number of modes used was presented.

When the first four mode shape derivatives were computed for the examples used, Nelson's method was the least computationally intensive, and since it is an exact method, it is the method recommended. When the original mode shapes were used as initial approximations to the subspace eigensolution of the perturbed problem, the finite-difference method was competitive with Nelson's method. The modified modal method always converged faster than the modal method when at least as many modes were used in the approximation as the number of the mode shape being differentiated. Data were presented to show that the modified modal method can compete with Nelson's method for the first mode shape derivative when the number of modes needed in the summation was known before the eigensolution was performed.

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